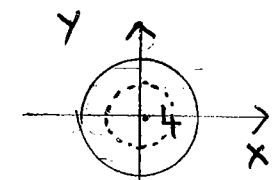
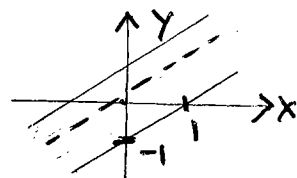


CALCULUS II TUTORIAL (7): SOLUTIONS

- ① (a) $f = \frac{1}{\sqrt{16-x^2-y^2}}$. Domain is $x^2+y^2 < 16 \rightarrow$ 
 (i.e. region inside circle of radius 4) where f is real and finite.
 Level curves are circles centred at origin with radius less than 4.
 Boundary is circle itself. Range is $[\frac{1}{4}, \infty)$ since $\sqrt{\quad}$ is positive root. Domain is bounded and open since circle excluded.

(b) $f = \ln(x^2+y^2)$. Domain is entire (x,y) plane except origin (since $\ln 0 = -\infty$). Range is $(-\infty, \infty)$. Only boundary point is $(0,0)$. Domain is open and unbounded. Level curves are circles centred at origin.

(c) $f = \sin^{-1}(y-x)$. Domain is $-1 \leq y-x \leq 1$ since $|\sin x| \leq 1$. Domain boundary $y-x = \pm 1$ is included since f defined there. Domain is closed since includes boundary. Unbounded since not covered by finite disc. Range is $[-\pi/2, \pi/2]$. Level curves are lines $y-x = c$ ($-1 \leq c \leq 1$)



- ② Level curve of $f = 16-x^2-y^2$ through $(2\sqrt{2}, \sqrt{2})$ is given by
 $16-x^2-y^2 = 16-(2\sqrt{2})^2-(\sqrt{2})^2 = 6 \Rightarrow x^2+y^2 = 10$ (circle)
 Level surface of $f = \sqrt{x-y} - \ln z$ through $(3, -1, 1)$ is
 $\sqrt{x-y} - \ln z = \sqrt{3+1} - \ln 1 = 2 \Rightarrow z = e^{\sqrt{x-y}-2}$ with $x \geq y$.

- ③ (a) $\lim_{\substack{(x,y) \rightarrow (1,1) \\ (x \neq y)}} \left(\frac{x^2-2xy+y^2}{x-y} \right) = \lim_{(1,1)} \frac{(x-y)^2}{(x-y)} = \lim_{(1,1)} (x-y) = 0$
 (b) $\lim_{\substack{(x,y) \rightarrow (2,0) \\ (2x-y \neq 4)}} \frac{\sqrt{2x-y}-2}{2x-y-4} = \lim_{(2,0)} \frac{\sqrt{2x-y}-2}{(\sqrt{2x-y}-2)(\sqrt{2x-y}+2)}$
 $= \lim_{(2,0)} \frac{1}{\sqrt{2x-y}+2} = \frac{1}{\sqrt{4-0}+2} = \frac{1}{4}$

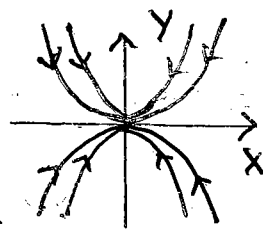
In both cases path to limit avoids region where denominator is zero.

- ④ (a) $f = \sin(x+y)$ is continuous everywhere
 (b) $f = \sin \frac{1}{xy}$ continuous unless $xy = 0 \Rightarrow$ lines $x=0, y=0$
 (c) $f = xy \sin \frac{1}{z}$ continuous unless $z=0$.

5

$$(a) \lim_{(x,y) \rightarrow (0,0)} \left[\frac{x^4 - y^2}{x^4 + y^2} \right] = \lim_{(0,0)} \left[\frac{x^4 - (kx^2)^2}{x^4 + (kx^2)^2} \right] = \frac{1-k^2}{1+k^2}$$

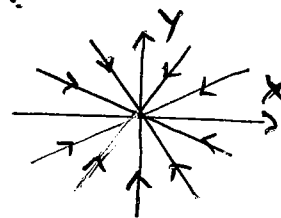
along parabola $y = kx^2$. Depends on $k \Rightarrow$ no limit



$$(b) \lim_{(x,y) \rightarrow (0,0)} \left[\frac{x-y}{x+y} \right] = \lim_{(0,0)} \left[\frac{x-mx}{x+mx} \right] = \frac{1-m}{1+m}$$

$(m \neq -1)$

along line $y = mx$. Depends on $m \Rightarrow$ no limit



6

$$(a) f = \frac{x^3 - xy^2}{x^2 + y^2} = \frac{r^3 \cos^3 \theta - r^3 \cos \theta \sin^2 \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = r \cos \theta \cos 2\theta$$

Hence $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{r \rightarrow 0} (r \cos \theta \cos 2\theta) = 0$ for all θ .

$$(b) f = \frac{y^2}{x^2 + y^2} = \frac{r^2 \sin^2 \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = \sin^2 \theta. \text{ Depends on } \theta \Rightarrow \text{no limit}$$

$$(c) f = \tan^{-1} \left(\frac{|x| + |y|}{x^2 + y^2} \right) = \tan^{-1} \left(\frac{r(|\cos \theta| + |\sin \theta|)}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} \right)$$

Hence $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{r \rightarrow 0} \tan^{-1} \left(\frac{|\cos \theta| + |\sin \theta|}{r} \right) = \tan^{-1} \infty = \frac{\pi}{2}$

7

$$(a) f = x + y + xy \Rightarrow f_x = \frac{\partial f}{\partial x} = 1 + y, f_y = \frac{\partial f}{\partial y} = 1 + x,$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 0, f_{yy} = \frac{\partial^2 f}{\partial y^2} = 0, f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = 1 = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = f_{yx}$$

$$(b) f = x^2 y + \cos y + y \sin x \Rightarrow f_x = 2xy + y \cos x,$$

$$f_y = x^2 - \sin y + \sin x, f_{xx} = \frac{\partial}{\partial x} (2xy + y \cos x) = 2y - y \sin x,$$

$$f_{xy} = \frac{\partial}{\partial y} (2xy + y \cos x) = 2x + \cos x, f_{yy} = \frac{\partial}{\partial y} (x^2 - \sin y + \sin x)$$

$$f_{yx} = \frac{\partial}{\partial x} (x^2 - \sin y + \sin x) = 2x + \cos x (= f_{xy}). \quad \underline{f_{yy} = -\cos y},$$

8

$$xy + z^3 x - 2yz = 0 \Rightarrow \frac{\partial}{\partial x} (xy + z^3 x - 2yz) = 0$$

$$\Rightarrow y + z^3 + 3z^2 x \frac{\partial z}{\partial x} - 2y \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{y + z^3}{2y - 3xz^2}$$

$$\text{Also } \frac{\partial}{\partial y} (xy + z^3 x - 2yz) = 0 \Rightarrow x + 3z^2 x \frac{\partial z}{\partial y} - 2z - 2y \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{x - 2z}{2y - 3xz^2}$$

$$\text{At } (1, 1, 1), \frac{\partial z}{\partial x} = \frac{1+1}{2-3} = -2, \frac{\partial z}{\partial y} = \frac{1-2}{2-3} = 1.$$